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2008 J. Phys. A: Math. Theor. 41 355301

(http://iopscience.iop.org/1751-8121/41/35/355301)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 355301 (9pp) doi:10.1088/1751-8113/41/35/355301

Quantum phase transitions in an effective Hamiltonian: fast and slow systems

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Received 23 May 2008, in final form 27 June 2008
Published 28 July 2008
Online at stacks.iop.org/JPhysA/41/355301

Abstract
An effective Hamiltonian describing interaction between generic *fast* and *slow* systems is obtained in the strong interaction limit. The result is applied for studying the effect of quantum phase transition as a bifurcation of the ground state of the *slow subsystem*. Examples such as atom–field and atom–atom interactions are analyzed in detail.

PACS numbers: 03.65.Fd, 42.50.Ct, 42.50.Hz

1. Introduction

Frequently, in the process of interaction between two quantum systems, only one of them can be detected experimentally. In this case, a variety of physical effects appear in the process of such interaction which can be described in terms of an effective Hamiltonian corresponding to the observed system. The simplest example of such a situation arises when a *fast system* interacts with a *slow system*. Then, the *fast system* can be adiabatically eliminated and the *slow system* is described by an effective Hamiltonian. These considerations were assumed in the famous Born–Oppenheimer approximation. A regular approach to the quantum dynamics of the observed system is provided by the Lie transformation method [1, 2]. The advantage of this method consists of the possibility of varying the system's parameters, changing relations between them, which allows us to describe different physical regimes using the same mathematical tool. In particular, such an important example as expansion on the resonances in quantum systems not preserving the number of excitations can be obtained [3]. In this case a generic Hamiltonian governing the interaction of two subsystems beyond the rotating wave approximation (RWA) can be represented as a series in operators describing all possible transitions in the system.

Several interesting features appearing in the process of interaction of quantum systems can be realized by studying evolution of only two generic quantum system with one quantum channel. Even in such a simple case we may discriminate at least three interesting limits: (a) when the interaction constant g is much higher than the characteristic frequencies of both interacting systems; (b) when g is much smaller than the frequencies of the systems and (c) when g is higher than or of the order of the frequency of one system but much smaller than the frequency of the other one.

Case (a) of very strong coupling should be studied carefully, because using the expansion parameter such as an interaction constant over a characteristic frequency could be quite tricky. For instance, the type of the spectrum corresponding to the non-perturbed and to the perturbed systems can be different: either continuous or discrete.

Case (b) corresponds to a situation where the resonance expansion is applicable. As was shown in [3], the evolution is governed by an effective Hamiltonian describing a certain resonant interaction and the representation space of the total system can always be divided into (almost) invariant subspaces.

The last case (c) possesses a peculiar property: besides finding a corresponding effective Hamiltonian, we can also project it out to the lower energy state of the fast system, which would never get excited under given relations between the system's parameters, and thus, describe an effective dynamics of the slow system in the limit of strong interaction. In this work we extend the Lie transformation method [2, 3] to this limit and show how the quantum phase transitions naturally emerge in this region of the system's parameters.

The quantum phase transitions (QPT) are a common feature of nonlinear quantum systems. Such transitions occur at zero temperature and are associated with an abrupt change in the ground-state structure. QPT are related to singularities in the energy spectrum and, at the critical points defining QPT, the ground-state energy is a non-analytic function of the system's parameters [5]. Qualitatively, for a wide class of quantum systems, several important properties of QPT can be studied in the thermodynamic [6] and semiclassical limits [7, 8]. Then, QPT can be analyzed in terms of a classical effective potential energy surface [9]. In this language QPT are related to the appearance of a new classical separatrix when the coupling parameters acquire certain values. According to the standard semiclassical quantization scheme and the correspondence principle, the energy density is proportional to the classical period of motion, diverging on the separatrix, which explains a high density of quantum states at the critical points.

Our goal is to study effective Hamiltonians describing the evolution of a generic quantum system X interacting with a quantum system Y in the case where the characteristic frequency of the system X is essentially lower than the corresponding frequency of the system Y, $\omega_X \ll \omega_Y$, and the interaction constant g satisfies the strong coupling condition: $\omega_X \lesssim g \ll \omega_Y$. We show that depending on the type of interaction and the nature of quantum systems different physical situations take place, but generically such effective Hamiltonians describe quantum phase transitions in the *slow system*.

2. Effective Hamiltonian

Let us consider the following generic Hamiltonian describing an interaction between two quantum systems:

$$H = \omega_1 X_0 + \omega_2 Y_0 + g(X_+ + X_-)(Y_+ + Y_-), \tag{1}$$

where X_0 and Y_0 are the free Hamiltonians of the X and Y systems respectively, and such that $\omega_1 \ll \omega_2$. The above Hamiltonian does not preserve the *total excitation number operator*

$N = X_0 + Y_0$ and, in the limit $\omega_1, \omega_2 \gg g$, leads to the appearance of *multiphoton-type* interactions of the form $X_+^n Y_-^m$ which, under certain physical conditions on the frequencies $\omega_{1,2}$, describe *resonant* transitions between energy levels of the whole system (see [3] and references therein).

The raising–lowering operators $X_\pm$, $Y_\pm$ describe transitions between energy levels of the systems X and Y respectively and consequently obey the following commutation relations:

$$[X_0, X_\pm] = \pm X_\pm, \qquad [Y_0, Y_\pm] = \pm Y_\pm.$$

We do not impose any condition on the commutators between transition operators, which are generally some functions of diagonal operators and of some integrals of motion $[N_1, X_0] = [N_2, Y_0] = 0$:

$$[X_+, X_-] = \nabla_{X_0}\phi_x(X_0, N_1),$$
$$[Y_+, Y_-] = \nabla_{Y_0}\phi_y(Y_0, N_2),$$

where $\phi_x(X_0, N_1) = X_+X_-$ and $\phi_y(Y_0, N_2) = Y_+Y_-$ are some polynomials of X_0 and Y_0 respectively (from now on we omit the dependence on integrals $N_{1,2}$ in the arguments) and $\nabla_z\phi(z) = \phi(z) - \phi(z+1)$. The objects $(X_0, X_\pm)$ and $(Y_0, Y_\pm)$ are known as polynomial deformed algebras $sl_{\mathrm{pd}}(2, R)$ [11].

Now, we will be interested in the limit where the *slow system* frequency is less than/or of the order of the coupling constant, $\omega_1 \lesssim g \ll \omega_2$. It is worth noticing that the rotating wave approximation cannot be directly applied to Hamiltonian (1) in this regime; this is because the contributions of the counter-rotating and rotating terms are of the same order. However, the Lie transformation method can be used following the method [2] we can adiabatically remove all the terms that contain the fast system's transition operators, $Y_\pm$. In particular, the counter-rotating term $X_+Y_+ + X_-Y_-$ and the rotating term $X_+Y_- + X_-Y_+$ can be eliminated from the Hamiltonian (1) by a subsequent application of the following Lie-type transformations:

$$U_1 = \exp[\varepsilon(X_+Y_+ - X_-Y_-)], \tag{2a}$$

$$U_2 = \exp[\epsilon(X_+Y_- - X_+Y_-)], \tag{2b}$$

where the small parameters, ε and ϵ, are defined by

$$\varepsilon = \frac{g}{\omega_2 + \omega_1} \ll 1, \qquad \epsilon = \frac{g}{\omega_2 - \omega_1} \ll 1.$$

The transformations (2*a*) and (2*b*) generate different kinds of terms: such as $X_\pm^n Y_\pm^k$ + h.c., $X_\pm^n Y_\mp^k$ + h.c., $Y_\pm^n$ + h.c., and $X_\pm^n$ + h.c. with coefficients depending on X_0 and Y_0. Under the condition $\omega_1, g \ll \omega_2$ all the rapidly oscillating terms, i.e. those containing powers of $Y_\pm$, can be removed by applying transformations similar to (2*a*), (2*b*), with properly chosen parameters. In particular, in this paper we will restrict ourselves by terms up to third order on the small parameters, which are sufficient to analyze the phenomena of our interest. Then, the transformation required to eliminate the terms that contain the transition operators $Y_\pm$, which appear after applying (2*a*), (2*b*), has the form

$$U_3 = \exp\left[\frac{\delta\varepsilon}{2}\left(Y_+^2 - Y_-^2\right)\nabla_x\phi_x(X_0)\right]. \tag{3}$$

The result can be expressed as a power series of the single parameter $\delta = g/\omega_2 \ll 1$.

It is worth noting that it is not enough that δ be a small parameter for the formal expansion in (2*a*) (and the subsequent transformations). A balance is necessary between the *effective dimensions* of the subsystems and δ. The *effective dimensions* of the system depend on the order of the polynomials $\phi_{x,y}$, and on the powers of the elements $X_{\pm,0}$ and $Y_{\pm,0}$ involved in

each transformation. It was shown before [3], that the powers of the small parameters are increasing faster than the powers of $X_{\pm,0}$ and $Y_{\pm,0}$, which implies that we can focus on the *effective dimensions* introduced with (2*a*).

Taking into account the above-mentioned considerations, keeping only terms up to third order in δ and disregarding small corrections to the effective transition frequencies and to the quadratic term, we arrive at the following effective Hamiltonian, diagonal (in our approximation) on the operators of the Y system,

$$H_{\text{eff}} = \omega_1 X_0 + \omega_2 Y_0 - 2\omega_1\delta^2\nabla_{x,-y}\Phi(X_0, Y_0+1) + g\delta\nabla_y\phi_y(Y_0)(X_+ + X_-)^2 + \tfrac{1}{2}g\delta^3\nabla_y\big(\phi_y(Y_0)\nabla_y^2\phi_y(Y_0-1)\big)(X_+ + X_-)^4, \tag{4}$$

where

$$\Phi(X_0, Y_0) = \phi_x(X_0)\phi_y(Y_0),$$

and the generalized displacement operators are defined as

$$\nabla_{mX_0,nY_0} f(X_0, Y_0) = f(X_0, Y_0) - f(X_0+m, Y_0+n),$$

for m and n integers.

Because the effective Hamiltonian (4) is diagonal for the operators of the *fast system* (Y), we may project it out onto a minimal energy eigenstate of the Y system, $|\psi_0\rangle_Y$, substituting Y_0 by its eigenvalue y_0: $Y_0|\psi_0\rangle_Y = y_0|\psi_0\rangle_Y$, where the parameter y_0 is usually directly related to the dimension of the system Y.

The first-order effect comes from the term $\sim (X_+ + X_-)^2$, while the term $\sim (X_+ + X_-)^4$ defines a *fine structure* of the *effective potential*, obtained after projecting the effective Hamiltonian (4) onto the state $|\psi_0\rangle_Y$.

It is easy to evaluate the effect of transformations (2*a*), (2*b*) and (3) on the average values of the system's observables. Since we are interested in applying this approach to QPT, we will study the ground-state effect. Thus, let us suppose that $|\varphi_0\rangle_X$ is the ground state of the effective Hamiltonian (4) projected onto the state $|\psi_0\rangle_Y$,

$$H_{\text{eff}}(y_0)|\varphi_0(y_0)\rangle_X = \varepsilon_0(y_0)|\varphi_0(y_0)\rangle_X.$$

Then, the ground state of the whole system is

$$|\Psi\rangle = U^\dagger|\varphi_0\rangle_X|\psi_0\rangle_Y, \tag{5}$$

where $U = U_3U_2U_1$, and correspondingly, any average value should be computed using $|\Psi\rangle$:

$$\langle L\rangle = \langle\Psi|L|\Psi\rangle = {}_Y\langle\psi_0|_X\langle\varphi_0|\tilde{L}|\varphi_0\rangle_X|\psi_0\rangle_Y, \qquad \tilde{L} = ULU^\dagger. \tag{6}$$

In particular, it is easy to evaluate $\langle Y_0\rangle$ and $\langle X_0\rangle$ up to second order on the small parameter δ:

$$\langle Y_0\rangle = y_0 - \delta_X^2\langle\varphi_0|(X_+ + X_-)^2|\varphi_0\rangle_X\nabla_y\phi_y(y_0), \tag{7}$$

$$\langle X_0\rangle = {}_X\langle\varphi_0|X_0|\varphi_0\rangle_X - \delta_X^2\langle\varphi_0|(X_+ + X_-)^2|\varphi_0\rangle_X\nabla_y\phi_y(y_0) + 2\delta^2({}_X\langle\varphi_0|\phi_x(X_0+1)|\varphi_0\rangle_X\phi_y(y_0) - {}_X\langle\varphi_0|\phi_x(X_0)|\varphi_0\rangle_X\phi_y(y_0+1)), \tag{8}$$

where the averages ${}_X\langle\varphi_0|(X_+ + X_-)^2|\varphi_0\rangle_X$ and ${}_X\langle\varphi_0|\phi_x(X_0)|\varphi_0\rangle_X$ clearly depend on the parameters δ and y_0.

It is important to stress that, although δ is a small parameter, the effect of the terms $\sim\delta^n$, $n \geqslant 1$, could be in principle comparable with the main diagonal term ω_1X_0, especially if the algebra of X operators describe a *big subsystem*, i.e., large spin or big photon number. In this case non-trivial effects such as QPT may occur.

Now, we may proceed with analysis of the effective Hamiltonian (4), focusing on the possible bifurcation of the ground state. The simplest way to study the critical properties of the effective Hamiltonian (4) consists of taking the thermodynamic limit, when after an appropriate rescaling of the interaction constants (see [14]) we tend the dimension of the fast system to infinity. It is worth noting that equations (5)–(8) allow us to study the finite size effect, i.e. corrections to average values of the system's observables for large, but finite values of the dimension of the system Y.

3. Examples

3.1. Atom–field interaction (Dicke model)

The Hamiltonian governing the evolution of A symmetrically prepared two-level atoms interacting with a single mode of a quantized field has the form [13]

$$H = \omega_1 \hat{n} + \omega_2 S_z + \frac{g}{\sqrt{A}}(S_+ + S_-)(a^\dagger + a), \tag{9}$$

where $\hat{n} = a^\dagger a$ and $S_{z,\pm}$ are generators of the $(A+1)$-dimensional representation of the $su(2)$ algebra. There are two important limit cases of the Dicke model: (a) when the field mode interacts with a collection of atoms, which transition frequencies are essentially larger than the field frequency; (b) when the atomic system interacts with a rapidly oscillating quantized field. In the first case the effective potential for the field (oscillator) mode may change its topology (when some conditions on the parameters of the system are satisfied) leading to the ground-state bifurcation and in particular, to a spontaneous 'condensation' of the virtual photons into 'real' field excitations. In the second case, the atomic system effectively behaves as the well-known Lipkin–Meshkov–Glick model, for which the ground state becomes degenerate for certain values of the parameters, leading to a macroscopic change of the ground-state energy.

3.1.1. Effective field dynamics. First let us suppose that the atoms form a *fast subsystem* so that

$$X_0 = \hat{n}, \qquad X_+ = a^\dagger, \qquad X_- = a, \qquad Y_0 = S_z, \qquad Y_\pm = S_\pm,$$

and thus, $\phi_y(Y_0) = C_2 - S_z^2 + S_z$ and $\phi_x(X_0) = \hat{n}$, where $C_2 = A/2(A/2+1)$ is the eigenvalue of the Casimir operator of the $su(2)$ algebra (integral of motion corresponding to the atomic subsystem).

Projecting the effective Hamiltonian onto the minimum energy state of the atomic system $|0\rangle_{\rm at}$, so that $y_0 = -A/2$, we obtain the following effective Hamiltonian for the field mode:

$$H_{\rm eff} = \tilde{\omega}_1 \hat{n} - g\delta(a + a^\dagger)^2 + gA^{-1}\delta^3(a + a^\dagger)^4 - \frac{A}{2}\omega_2, \tag{10}$$

where $\tilde{\omega}_1 = \omega_1(1 - 2\delta^2)$.

Rewriting (10) in terms of position and momentum operators,

$$H_{\rm eff} = \frac{\tilde{\omega}_1}{2}(p^2 + x^2) - 2g\delta x^2 + 4gA^{-1}\delta^3 x^4 - \frac{A}{2}\omega_2,$$

we immediately detect that QPT in this case is related to the bifurcation of the effective potential $U(x) = (\tilde{\omega}_1/2 - 2g\delta)x^2 + 4gA^{-1}\delta^3 x^4$ (up to a constant shift) from a single minimum at $x = 0$ for $4g\delta < \tilde{\omega}_1$ to a symmetric double well structure at $4g\delta > \tilde{\omega}_1$, with minima at $x_* = \pm\sqrt{\Delta A/(16\delta^3)}$, where $\Delta = 4\delta - \tilde{\omega}_1/g > 0$. We stress that the above effective potential describes the system behavior only in the weak excitation limit, $\Delta \ll \omega_1/g$.

The physical effect associated with this QPT consists of a spontaneous generation of photons in the field mode. In some sense, the virtual photons, always presented in the Dicke model (9), are *condensed* into the real photons after crossing the critical point $4g\delta = \tilde{\omega}_1$.

The finite size effect, can be appreciated from equations (7) and (8). For instance, for the average value of the atomic inversion operator, we obtain

$$\langle S_z \rangle_f \approx -\frac{A}{2}\left(1 - 4\frac{\delta^2}{A}\langle\varphi_0|\hat{q}^2|\varphi_0\rangle_f\right),$$

where $|\varphi_0\rangle_f$ is the solution of the equation

$$\left[\hat{p}^2 - \Delta\frac{g}{\tilde{\omega}_1}\hat{q}^2 + 8\frac{g}{A\tilde{\omega}_1}\delta^3\hat{q}^4\right]|\varphi_0\rangle = \left(\frac{2E_0}{\tilde{\omega}_1} + \frac{A\omega_2}{\tilde{\omega}_1}\right)|\varphi_0\rangle,$$

where $\hat{q} = (a + a^\dagger)/\sqrt{2}$ and $\hat{p} = (a - a^\dagger)/\mathrm{i}\sqrt{2}$, and E_0 is the ground-state energy, which can be obtained in the frame of the standard perturbation theory in the limit $\Delta \to 0$, allowing to estimate $\langle\varphi_0|\hat{q}^2|\varphi_0\rangle_f$ by using the Feynman–Hellman theorem, taking as independent parameters, for instance A and g, so that $\Delta = \Delta(g)$ and $\delta = \delta(g)$.

In the thermodynamic limit ($A \to \infty$) we obviously have

$$\frac{1}{A}\langle\varphi_0|\hat{q}^2|\varphi_0\rangle \approx \begin{cases} 0, & 4g\delta \leqslant \tilde{\omega}_1 \\ x_*^2/A = \Delta/(16\delta^3), & 4g\delta \geqslant \tilde{\omega}_1, \end{cases}$$

so that [6] $\langle S_z\rangle/A \approx -\tilde{\omega}_1\omega_2/(4g^2)$.

We can also easily evaluate, in the leading order on Δ, the average number of photons created in the field (normalized by the number of atoms) after passing the phase transition point (superradiant phase), which in the thermodynamic limit has the form

$$\frac{\bar{n}}{A} \approx \frac{1}{2A}\langle\varphi_0|\hat{p}^2 + \hat{q}^2|\varphi_0\rangle \approx \frac{1}{2A}x_*^2 = \frac{\Delta}{32\delta^3} \approx \frac{\Delta\omega_2^2}{8g\omega_1},$$

and coincides with the one obtained in [6] in our approximation.

3.1.2. Effective atomic dynamics. In the opposite case, when the atoms form a *slow subsystem* we have

$$X_0 = S_z, \qquad X_\pm = S_\pm, \qquad Y_0 = \hat{n}, \qquad Y_+ = a^\dagger, \qquad Y_- = a.$$

Projecting the effective Hamiltonian onto the minimum energy state of the field mode $|0\rangle_f$, so that $y_0 = 0$, the effective Hamiltonian acquires the form

$$H_{\mathrm{eff}} = \tilde{\omega}_1 S_z - 4\frac{g\delta}{A}S_x^2 + 2\frac{\omega_1\delta^2}{A}S_z^2, \tag{11}$$

where $\tilde{\omega}_1 = \omega_1 - 2\omega_2\delta^2/A$.

For our analysis it is convenient to perform a $\pi/2$ rotation in (11) around axis y (this avoids the coincidence of the physically important south pole of the sphere with the singular point), transforming the Hamiltonian (11) into

$$\tilde{H}_{\mathrm{eff}} = -\tilde{\omega}_1 S_x - 4\frac{g\delta}{A}S_z^2 + 2\frac{\omega_1\delta^2}{A}S_x^2. \tag{12}$$

In the semiclassical limit we may replace the atomic operators by the corresponding classical vectors over the two-dimensional sphere, i.e.,

$$S_z \to \frac{A}{2}\cos\theta, \qquad S_x \to \frac{A}{2}\sin\theta\cos\phi, \qquad S_y \to \frac{A}{2}\sin\phi\sin\theta,$$

and thus rewrite the effective Hamiltonian (12) as a classical Hamiltonian function,

$$H_{\rm cl} = -\frac{A}{2}(\tilde{\omega}_1 \cos\phi \sin\theta + 2g\delta \cos^2\theta - \omega_1\delta^2 \cos^2\phi \sin^2\theta). \tag{13}$$

The first two terms in the above expression describe the semiclassical limit of the Lipkin–Meshkov–Glick model [8] and determine the critical point of QPT, $\xi = 4g\delta/\tilde{\omega}_1 = 1$, which again is related to the bifurcation of the ground state: a single minimum at $\sin\phi = 0$, $\cos\theta_* = 0$ splits into two minima at $\sin\phi = 0$, $\cos\theta_{**} = \pm\sqrt{1-\xi^{-2}}$ for $\xi > 1$. It is worth noting that the global minimum of $H_{\rm cl}$ at $\xi < 1$ converts into a local maximum for $\xi > 1$, so that [8]

$$H_{\rm cl}(\theta_{**}) \approx -\frac{A}{4}(\xi + \xi^{-1}) < H_{\rm cl}(\theta_*) = -\frac{A}{2}.$$

This means that the atoms, initially prepared at the minimum of the Hamiltonian function, spontaneously change their ground-state energy at some value of the system's parameters. Classically, this implies the appearance of a separatrix, which leads to the discontinuity on the energy density spectrum in the semiclassical limit. It is also worth noting that there is a loss of the rotational symmetry in this process: the new ground state is obviously not invariant under rotations around axis x, while the initial ground state is clearly invariant under x rotations.

It is easy to see that the last term in (13) is of lower order in the parameter δ and can be neglected in the first approximation for description of QPT at $\xi = 1$.

The finite size effect, can be appreciated from equation (7), leading for instance for the following expression for the average photon number,

$$\frac{\langle\hat{n}\rangle_{\rm a}}{A} \approx 4\frac{\delta^2}{A^2}\langle S_z^2\rangle_{\rm a},$$

where $\langle\rangle_a$ means the average over the atomic state $|\varphi_0\rangle_a$, which is the ground state of the Hamiltonian

$$\tilde{H}_{\rm eff} = -\tilde{\omega}_1 S_x - 4\frac{g\delta}{A}S_z^2.$$

Again, in the thermodynamic limit ($A \to \infty$) one obtains

$$\frac{\langle\hat{n}\rangle}{A} \approx \begin{cases} 2\delta^2\left(1 - \left(\dfrac{\tilde{\omega}_1}{4g\delta}\right)^2\right) & \xi \geqslant 1 \\ 0, & \xi \leqslant 1. \end{cases}$$

3.2. Spin–spin interaction

As a second example let us consider a dipole–dipole-like interaction, that is,

$$H = \omega_1 S_{z1} + \omega_2 S_{z2} + g S_{x1} S_{x2},$$

so that $X_0 = S_{z1}$, $X_\pm = S_{\pm 1}$, $Y_0 = S_{z2}$, $Y_\pm = S_{\pm 2}$. The effective Hamiltonian for the *slow spin system* (after projecting onto the lowest state of the *fast spin system* with eigenvalue $-A_2/2$) takes the form similar to (11), with

$$H_{\rm eff} = \tilde{\omega}_1 S_{z1} - 2A_2 g\delta S_{x1}^2 + 2A_2\omega_1\delta^2 S_{z1}^2 + 16 g A_2 \delta^3 S_{x1}^4, \tag{14}$$

where $\tilde{\omega}_1 = \omega_1 - 2A_1\omega_1\delta^2$. The first two terms are dominant for $\delta \ll 1$ and describe the Lipkin–Meshkov–Glick model, so that the critical point is reached at $\xi = 4A_2A_1^2 g\delta/\tilde{\omega}_1 = 1$ in the thermodynamical limit. The effect of the rest of the terms is negligible in the vicinity of $\xi = 1$. The main difference between the above Hamiltonian, and the Hamiltonian (11) is the last term; in this case we obtain a S_{x1}^4 term because the algebra of the *fast* leads to a second order polynomial on Y_0, similar to the Hamiltonian (10) [3].

4. Conclusions

We deduce the effective Hamiltonian of a generic slow quantum system interacting with another fast oscillating system when the total excitation number is not preserved. Analyzing those effective Hamiltonians in the semiclassical limit we have observed a bifurcation of the ground state leading to the effect of the quantum phase transitions. We were interested exclusively in the intermediate regime, where $\omega_X \lesssim g \ll \omega_Y$, when the rotating wave approximation cannot be directly applied (because the contribution of the anti-rotating and rotating terms in the Hamiltonian is of the same order) and our main goal is to show that the Lie transformation method provides a useful tool for approaching nonlinearly interacting quantum systems.

It is interesting to note that, for multidimensional systems, when algebraically the X system is a direct sum of several non-interacting subsystems, an interesting effect of generation of entangled states (in the non-preserving excitation case) can be observed. Really, let us suppose that in (1) $X_{0,\pm} = X_{0,\pm 1} + X_{0,\pm 2}$, $[X_{j,1}, X_{j,2}] = 0$, $j = 0, \pm$; then the corresponding effective Hamiltonian (up to a first non-trivial order in δ) takes the form

$$H_{\text{eff}} \approx \omega_1(X_{0,1} + X_{0,2}) + \omega_2 Y_0 + g\delta[(X_{+,1} + X_{-,1})^2 + (X_{+,2} + X_{-,2})^2 + 2(X_{+,1} + X_{-,1})(X_{+,2} + X_{-,2})]\nabla_y \phi_y(Y_0),$$

where we can clearly see that the last term contains the operator product $\sim X_{+,1}X_{+,2}$ which, together with quadratic terms in $X_{\pm,1(2)}$, implies a spontaneous generation of entangled states of X_1 and X_2 starting from the minimum energy state. This can be corroborated by the entangling power measure by considering a uniform distribution of the initial factorized states [12]. Thus we can say that, in the regime studied here, entanglement can be generated in a bipartite system the vicinity of a phase transition.

Acknowledgments

One of the authors (I Sainz) thanks STINT (Swedish Foundation for International Cooperation in Research and Higher Education) for support. This work was supported by grants: CONACyT 45704, Milenio ICM P06-067-F and FONDECyT 1080535.

References

[1] Steinberg S 1987 *Lie Methods in Optics (Lecture Notes in Physics* vol 250*)* (Berlin: Springer) p 45
[2] Klimov A B and Sánchez-Soto L L 2000 *Phys. Rev.* A **61** 063802
Klimov A B, Sánchez-Soto L L, Navarro A and Yustas E C 2002 *J. Mod. Opt.* **49** 2211–26
Sainz I, Klimov A B and Roa L 2006 *Phys. Rev.* A **73** 032303
[3] Klimov A B, Sainz I and Chumakov S M 2003 *Phys. Rev.* A **68** 063811
Roa L, Krügel A and Saavedra C 2007 *Phys. Lett.* A **366** 563–8
[4] Sachdev S 1999 *Quantum Phase Transitions* (Cambirdge: Cambridge University Press)
[5] Iachello F and Zamfir N V 2004 *Phys. Rev. Lett.* **92** 212501
[6] Emary C and Brandes T 2003 *Phys. Rev. Lett.* **90** 044101
Emary C and Brandes T 2003 *Phys. Rev.* E **67** 066203
[7] Heiss W D, Scholtz F G and Geyer H B 2005 *J. Phys. A: Math. Gen.* **38** 1843–51
Dusuel S and Vidal J 2005 *Phys. Rev.* B **71** 224420
Zanardi P and Paunkovic N 2006 *Phys. Rev.* E **74** 031123
[8] Lipkin H J, Meshkov N and Glick A J 1965 *Nucl. Phys.* **62** 188–98
Botet R and Jullien R 1983 *Phys. Rev.* B **28** 3955–3967
Cirac J I, Lewenstein M, Mølmer K and Zoller P 1998 *Phys. Rev.* A **57** 1208–1218
Leyvraz F and Heiss W D 2005 *Phys. Rev. Lett.* **95** 050402
Castaños O, López-Peña R, Hirsch J G and López-Moreno E 2006 *Phys. Rev.* B **74** 104118
[9] Liberti G, Plastina F and Piperno F 2006 *Phys. Rev.* A **74** 022324

[10] Milonni P W, Ackerhalt J R and Galbraith H W 1983 *Phys. Rev. Lett.* **50** 966–9
Zaheer K and Zubairy M S 1988 *Phys. Rev.* A **37** 1628–1633
[11] Higgs P W 1979 *J. Phys. A: Math. Gen.* **12** 309–23
Roĉek M 1991 *Phys. Lett.* B **255** 554–7
Bonatsos D, Daskaloyannis C and Lalazissis G A 1993 *Phys. Rev.* A **47** 3448–3451
Karassiov V P 1992 *J. Sov. Laser Res.* **13** 188
Karassiov V P 1994 *J. Phys. A: Math. Gen.* **27** 153–65
Karassiov V P and Klimov A B 1994 *Phys. Lett.* A **189** 43–51
Quesne C 1995 *J. Phys. A: Math. Gen.* **28** 2847–60
Abdesselam B, Beckers J, Chakrabarti A and Debergh N 1996 *J. Phys. A: Math. Gen.* **29** 3075–88
Sunilkumar V, Bambah B A, Jagannathan R, Panigrahi P K and Srinavasan V 2000 *J. Opt. B: Quantum Semiclass. Opt.* **2** 126–32
[12] Zanardi P, Zalka C and Faoro L 2000 *Phys. Rev.* A **62** 030301
Wang X and Zanardi P 2002 *Phys. Rev.* A **66** 044303
[13] Dicke R H 1954 *Phys. Rev.* **93** 99–110
[14] Zhang W M, Feng D H and Gilmore R 1990 *Rev. Mod. Phys.* **62** 867–927